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"Applications of Term Structure Models to Commodity Markets"

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In the last couple of decades there has been a wealth of new instruments in the financial markets. Among these new instruments we find the *derivatives*, a class that has deserved a lot of "publicity". In this note we describe some of the methods underlying the pricing of commodity derivatives.

1 Introduction. Derivatives

Among the financial instruments we can distinguish two large sets. The first one being the "underlying" securities such as stocks, bonds, currencies, commodities. The second set is the set of the instruments that are derived from the underlying ones. Derivatives are used to reduce risk, i.e. to hedge or simply to speculate. Their flexibiliy allow hedgers to fix a price in the future but also it allows investors and speculators to create complicated payoffs and to leverage positions.

Ever since Fisher Black and Myron Scholes published their seminal paper in 1973 new applications have not ceased to multiply.

Maybe the most used of the derivatives nowadays are financial futures and options. A *futures contract* is an agreement between two parties to interchange an asset in the future at a prefixed price. The person who buys the asset is said to be *long* the contract and, clearly, he or she will benefit in the case that the asset goes up in value. It is also clear that the potential loss amounts to the prearranged price. An *option* is like a future but with downside limited by zero.

The paper is organized as follows. In the next secion we describe the basic assumptions in the classic Black and Scholes model. In section 3 we deduce the Black and Scholes partial differential equation. Section 4 is devoted to examples of volatility curves from crude oil and natural gas. In section 5 we give a first approach to modeling a term structure and we give details of a translation of the model of F. Black and P. Karasinski (originally developed for interest rates). We move to the description of multiple factors and Heath-Jarrow-Morton in section 6 in where we, also, provide insight on why there is fundamental statistical similarities in the fixed income and commodity markets.

2 Black-Scholes-Merton World

Suppose that we have an asset S which is riskless and whose annualized percentage growth is μ.We can express its dynamics by the *ode* given by:

$$\frac{dS}{S} = \mu dt.$$

As we know assets are not riskless so we need to come up with a model that accounts for the risk. The first assumption we make is that the random variable representing the percentage change in the value of the asset follows a normal distribution with mean 0 and variance "proportional" to the risk we want to model. In discrete terms

$$\frac{S_{t+h} - S_t}{S_t} = \mu h + \phi_t \qquad (1)$$

where $\phi_t \sim N(0, \sigma^2 h)$. Another assumption we will make is to assume that ϕ_t and ϕ_{t+h} are independent. A natural question that arises at this point is that of the meaning of equation (1) if we wanted to go from a discreet to a continuous formulation.

2.1 Brownian Motion

A Brownian Motion is a stochastic process W that satisfies

1) $W(t)$ is continuous with probability 1.
2) $W(t) - W(s) \sim N(0, t-s)$.
3) $W(t) - W(s)$ is independent from $W(s)$.

As we can see 2) and 3) are the same conditions assumed for equation (1). Therefore we can rewrite (1) in a continuous setting as

$$\frac{dS}{S} = \mu dt + \sigma dW. \qquad (2)$$

Remarks: 1) From condition 2) we get that $W(t+h) - W(t) \sim \sqrt{h}N(0,1)$ and then $E(h(N(0,1))^2) = h$. This suggests that $(dW)^2 \sim dt$. This important fact is the key difference between usual and stochastic calculus. We will come back to this shortly.

2) σ, assumed to be constant, is called the **volatility** of S.

2.2 Solving the Equation for S. Itô's Lemma

We would like to be able to solve (2). Formally, we can apply Taylor's expansion to $\log(S)$ to get

$$\log S - \log S_0 = \frac{1}{S_0}\underbrace{(S - S_0)}_{dS} - \frac{1}{2S_0^2}\underbrace{(S - S_0)^2}_{dS^2} \qquad (3)$$

Still formally,

$$(dS)^2 = S^2\mu^2(dt)^2 + S^2\sigma^2(dW)^2 + 2\mu\sigma S^2 dt dW \tag{4}$$

If we now discard any term bigger than dt we obtain, using the previous remark, that $(dS)^2 \sim \sigma^2 dt$. We just need to plug in into (3) to obtain the formula for S:

$$S = S_0 e^{(\mu - \frac{\sigma^2}{2})dt + \sigma dW} \tag{5}$$

3 Valuation of Options

As we have said before, an option is the right (not obligation) to buy or sell an asset at a previously specified price K (called the strike). Therefore there are two main types:

1) Call Option: Gives the holder the right to buy the asset. Its final payoff is C(S)=max(S-K,0).

2) Put Option: Gives the holder the right to sell the asset. Its final payoff is P(S)=max(K-S,0).

We can do the same procedure as we did before with $\log(S)$ (which is nothing more than Itô's Lemma) to find the stochastic equation that, for example, C(S) satisfies.

$$dC = \sigma S \frac{\partial C}{\partial S} dW + (\mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t})dt \tag{6}$$

We now form the portfolio Π containing one call option and $-\Delta$ (to be determined) shares:

$$d\Pi = dC - \Delta dS$$

From (2) and (6):

$$d\Pi = (\mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - \mu \Delta S)dt + \sigma S(\frac{\partial C}{\partial S} - \Delta)dW.$$

By choosing $\Delta = \frac{\partial C}{\partial S}$ we see that we eliminate all risk, and therefore Π has to satisfy the equation corresponding to a riskless asset:

$$d\Pi = \Pi r dt = (C - \frac{\partial C}{\partial S} S) r dt = \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - \mu \Delta S)dt$$

and rearranging we obtain

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{7}$$

which is the Black-Scholes-Merton equation. We can get the price of calls and puts by solving the Black-Scholes-Merton equation with final conditions $\max(S-K,0)$ (call struck at K) and $\max(K-S,0)$ (put struck at K).

Remarks: 1) The only parameter which is not observable from the market is σ, there is no reference to μ. This leads to the subjects of **risk neutral valuation** and **delta hedging**.

2) The value of σ that matches the price in the market is called **implied volatility**.

3) Options with different strikes could (and actually they do) trade at different implied volatilities. This phenomenon clearly violates our hypothesis. The curve describing the implied volatility as a function of the strike is called the **smile**.

4 Examples.

4.1 Crude Oil

In the US crude oil market there are monthly future contracts. Each future contract matures around the 20th of the month. Options are traded on each of the contracts and, therefore, we can back out their corresponding implied volatilities. Now, what should these volatilities be if were living in a B-S-M world? It can be proved that, given an asset S and a future contract (on S) $F(T)$, the volatility of $F(T)$ must be equal to the volatility of S. As we can see in the following figure, this dependence does not hold in reality.

4.2 Natural Gas. Seasonality

A similar phenomenon happens in the natural gas market. In this case the vol curve shows, also, a seasonal pattern.

5 Term Structure Models: First Approach

The examples from the crude oil and natural gas markets indicate that we can not just assume B-S-M dynamics for the cash markets and then deduce the futures dynamics from there. Two types of solutions have been proposed to solve this problem. The first one is to modify the basic dynamics for the cash variable S (equation (2)) so that futures volatilities match what it is observed in the market. We will describe a model originally proposed by Fisher Black and Piotr Karasinski which incorporates the concept of **mean reversion**.

5.1 Black-Karasinski

We start by describing the dynamics of the spot process,

$$d\log S_t = \alpha(f(t) - \log S_t)dt + \sigma_t dw_t \tag{8}$$

where α represents the speed of mean reversion and $f(t)$ is chosen so that the current term structure of forward prices gets reproduced by the model. We could also match the initial term structure of the volatilities, at this stage we prefer to assume a constant spot vol ($\sigma_t = \sigma$) and match the forward volatilities from there. We will have more to say about this in what follows.

To find a solution we multiply both sides by the integrating factor $e^{\alpha t}$ and obtain

$$d\log S_t e^{\alpha t} = \alpha(f(t) - \log S_t)e^{\alpha t}dt + \sigma_t e^{\alpha t}dw_t$$

$$d\log S_t e^{\alpha t} + \alpha\log S_t e^{\alpha t}dt = \alpha f(t)dt + \sigma_t e^{\alpha t}dw_t$$

$$d(e^{\alpha t}\log S_t) = \alpha f(t)dt + \sigma_t e^{\alpha t}dw_t$$

$$\int_t^T d(e^{\alpha s}\log S_s) = \int_t^T \alpha f(s)ds + \int_t^T \sigma_t e^{\alpha s}dw_s$$

$$e^{\alpha T}\log S_T - e^{\alpha t}\log S_t = \alpha\int_t^T f(s)ds + \int_t^T \sigma_s e^{\alpha s}dw_s$$

$$\log S_T = e^{-\alpha(T-t)}\log S_t + e^{-\alpha T}\alpha\int_t^T e^{\alpha s}f(s)ds + e^{-\alpha T}\int_t^T \sigma_s e^{\alpha s}dw_s$$

$$S_T = e^{e^{-\alpha(T-t)}\log S_t + e^{-\alpha T}\alpha\int_t^T e^{\alpha s}f(s)ds}e^{e^{-\alpha T}\int_t^T \sigma_s e^{\alpha s}dw_s} \quad (9)$$

5.1.1 The Distribution of the Spot Process

Now, if we assume that $\sigma_t \sim \sigma$, we see that the random term involves $\int_t^T e^{\alpha s}dw_s$. To deal with it we make use of the technique called change of time. By setting

$$v(s) = e^{\alpha s}, c(s) = v^2(s) = e^{2\alpha s}, \beta_t = \int_0^t e^{2\alpha s}ds = \frac{e^{2\alpha t}-1}{2\alpha}$$

and

$$\alpha_t = inf\{s : \frac{e^{2\alpha s}-1}{2\alpha} > t\}$$

we obtain that

$$Y_t = \int_0^{\frac{\log(2\alpha t+1)}{2\alpha}} e^{\alpha s}dw_s$$

is a Brownian Motion, and therefore

$$\int_t^T e^{\alpha s}dw_s = \int_{\frac{\log(2\alpha(\frac{e^{2\alpha t}-1}{2\alpha})+1)}{2\alpha}}^{\frac{\log(2\alpha(\frac{e^{2\alpha T}-1}{2\alpha})+1)}{2\alpha}} e^{\alpha s}dw_s \sim N(0, \sqrt{\frac{e^{2\alpha T}-1}{2\alpha} - \frac{e^{2\alpha t}-1}{2\alpha}}). \quad (10)$$

We can now return to (9) to notice the distributional properties of S_T

$$\log S_T \sim N(e^{-\alpha(T-t)}\log S_t + \alpha e^{-\alpha T}\int_t^T e^{\alpha s}f(s)ds, \sigma e^{-\alpha T}\sqrt{\frac{e^{2\alpha T}-e^{2\alpha t}}{2\alpha}}) \quad (11)$$

5.1.2 The Distribution of the Forward Process

From (9), (10) and properties of lognormal distributions we get

$$F(t,T) = E_t(S_T) = S_t^{e^{-\alpha(T-t)}} e^{e^{-\alpha T}\alpha \int_t^T e^{\alpha s} f(s)ds} e^{\frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(T-t)})} \tag{12}$$

For simplicity, we rewrite (12) as

$$F(t,T) = E_t(S_T) = S_t^{e^{-\alpha(T-t)}} C_{t,T} \tag{13}$$

where

$$C_{t,T} = e^{e^{-\alpha T}\alpha \int_t^T e^{\alpha s} f(s)ds} e^{\frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(T-t)})} \tag{14}$$

As seen from time 0, the only random component of $F(t,T)$ is S_t. Therefore, knowing the distribution of S_t allows us to infer the dostribution of $F(t,T)$. From (12) and (11) with $T = t$ and $t = 0$

$$F(t,T) = e^{e^{-\alpha(T-t)}(e^{-\alpha t}\log S_0 + \alpha e^{-\alpha t}\int_0^t e^{\alpha s} f(s)ds + e^{\sigma e^{-\alpha t}\sqrt{\frac{e^{2\alpha t}-1}{2\alpha}} N(0,1)})} C_{t,T}$$

If we call $\overline{\mu}_{t,T}$ and $\overline{\sigma}_{t,T}$ to the mean and standard deviation in (11) we see that

$$S_t^{e^{-\alpha(T-t)}} \sim \text{lognormal}(e^{-\alpha(T-t)}\overline{\mu}_{0,t}, e^{-\alpha(T-t)}\overline{\sigma}_{0,t})$$

Therefore

$$\log F(t,T) \sim N(\log C_{t,T} + e^{-\alpha(T-t)}\overline{\mu}_{0,t}, e^{-\alpha(T-t)}\overline{\sigma}_{0,t}) \tag{15}$$

5.1.3 Matching the Current Term Structure

The function $f(t)$ is chosen in such a way so we match the initial forward curve. For that we set $F(0,T) = E(S_T)$.
Setting $t = 0$ and solving for $f(s)$ we get

$$\int_0^T e^{\alpha s} f(s)ds = \frac{\log(F(0,T)e^{-\frac{\sigma^2}{4\alpha}(1-e^{-2\alpha T})} S_0^{e^{-\alpha T}})}{\alpha e^{-\alpha T}} \tag{16}$$

The important point to notice here is that in order to calibrate the model we do not need to solve for $f(s)$, all we need is $\int_0^T e^{\alpha s} f(s)ds$.
We can now write the processes for S_t and for $F(t,T)$ in terms of the initial forward curve only

$$S_t = F(0,t)e^{-\frac{\sigma^2}{4\alpha}(1-e^{-2\alpha t})} e^{\sigma e^{-\alpha t}\sqrt{\frac{e^{2\alpha t}-1}{2\alpha}} N(0,1)}. \tag{17}$$

And by writing,

$$F(t,T) = e^{e^{-\alpha(T-t)}(\log F(0,t) - \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha t}))} e^{\alpha e^{\alpha T}\int_t^T e^{\alpha s} f(s)ds} \quad \text{x}$$

$$e^{\frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(T-t)})}e^{\sigma e^{-\alpha T}\sqrt{\frac{e^{2\alpha t}-1}{2\alpha}}N(0,1)}$$

we obtain

$$F(t,T) = e^{\log(F(0,T))+\frac{\sigma^2}{4\alpha}(e^{-2\alpha T}-e^{-2\alpha(T-t)})}e^{\sigma e^{-\alpha T}\sqrt{\frac{e^{2\alpha t}-1}{2\alpha}}N(0,1)}. \tag{18}$$

This says that $F(t,T) \sim \text{lognormal}(\mu^*_{t,T}, \sigma^*_{t,T})$ where

$$\mu^*_{t,T} = \log(F(0,T)) + \frac{\sigma^2}{4\alpha}(e^{-2\alpha T} - e^{-2\alpha(T-t)}) \tag{19}$$

$$\sigma^*_{t,T} = \sigma e^{-\alpha T}\sqrt{\frac{e^{2\alpha t}-1}{2\alpha}} \tag{20}$$

5.1.4 Volatility of the Spot Process

$$\log(S_{t_2}/S_{t_1}) = (e^{-\alpha(t_2-t_1)}-1)\log S_{t_1} + \alpha e^{-\alpha t_2}\int_{t_1}^{t_2} e^{\alpha s}f(s)ds + \sigma e^{-\alpha t_2}\int_{t_1}^{t_2} e^{\alpha s}dw_s$$

But we know that

$$\int_{t_1}^{t_2} e^{\alpha s}dw_s \sim N(0, \sqrt{\frac{e^{2\alpha t_2}-e^{2\alpha t_1}}{2\alpha}})$$

so

$$\text{stdev}(\log(S_{t_2}/S_{t_1})) = \sigma\sqrt{\frac{1-e^{-2\alpha(t_2-t_1)}}{2\alpha}} = \sigma e^{-\alpha(t_2-t_1)}\sqrt{\frac{e^{2\alpha(t_2-t_1)}-1}{2\alpha}}$$

If $t_2 \searrow t_1$ then $\sigma\sqrt{\frac{1-e^{-2\alpha(t_2-t_1)}}{2\alpha}} \searrow \sigma\sqrt{t_2-t_1}$ which says that the instantaneous vol at t_1 is σ.

5.1.5 Volatility of the Forward Process

From (13) we can write

$$\log(F(t_2,T)/F(t_1,T)) = \log(\frac{S_{t_2}^{e^{-\alpha(T-t_2)}}C_{t_2,T}}{S_{t_1}^{e^{-\alpha(T-t_1)}}C_{t_1,T}})$$

$$= \log(\frac{(S_{t_1}^{e^{-\alpha(T-t_1)}} const.e^{\sigma e^{-\alpha t_2}\int_{t_1}^{t_2}e^{\alpha s}dw_s})^{e^{-\alpha(T-t_2)}}C_{t_2,T}}{S_{t_1}^{e^{-\alpha(T-t_1)}}C_{t_1,T}})$$

$$= \log(const.e^{\sigma e^{-\alpha T}\int_{t_1}^{t_2}e^{\alpha s}dw_s})$$

Therefore,

$$\text{stdev}(\log(F(t_2,T)/F(t_1,T))) = \sigma e^{-\alpha(T-t_1)}\sqrt{\frac{e^{2\alpha(t_2-t_1)}-1}{2\alpha}} \tag{21}$$

If $t_2 \searrow t_1$ then $\sigma e^{-\alpha(T-t_1)}\sqrt{\frac{e^{2\alpha(t_2-t_1)}-1}{2\alpha}} \searrow \sigma e^{-\alpha(T-t_1)}\sqrt{t_2-t_1}$ which says that the instantaneous vol at t_1 is $\sigma e^{-\alpha(T-t_1)}$.

5.1.6 Limits as $\alpha \to 0$

We can recover the Black-Scholes-Merton world by letting $\alpha \to 0$. From (17) and (18) we obtain

$$\lim_{\alpha \to 0} S_t = F(0,t)e^{-\frac{\sigma^2}{2}(t)}e^{\sigma\sqrt{t}N(0,1)}$$

(notice that $F(0,t) = S_0 e^{(r-q)t}$).
Similarly,

$$\lim_{\alpha \to 0} F(t,T) = F(0,T)e^{-\frac{\sigma^2}{2}(t)}e^{\sigma\sqrt{t}N(0,1)}.$$

In the classical case we have

$$F(0,T) = S_0 e^{(r-q)T} \text{ ,and } F(t,T) = S_t e^{(r-q)(T-t)},$$

$$S_t = S_0 e^{(r-q-\frac{\sigma^2}{2})t}e^{\sigma\sqrt{t}N(0,1)},$$

and then

$$F(t,T) = S_0 e^{(r-q)T}e^{-\frac{\sigma^2}{2}t}e^{\sigma\sqrt{t}N(0,1)}.$$

5.1.7 Finding σ and α

From (21) we know that the annualized vol of $F(0,T)$ is given by

$$\sigma_{F(0,T)} = \sigma e^{-\alpha T}\sqrt{\frac{e^{2\alpha T}-1}{2\alpha}}\frac{1}{\sqrt{T}}$$

From where we can get

$$\sigma = \sigma_{F(0,T)}e^{\alpha T}\sqrt{\frac{2\alpha}{e^{2\alpha T}-1}}\sqrt{T} = \sigma_{F(0,T)}\sqrt{\frac{2\alpha T}{1-e^{-2\alpha T}}}$$

So, from market data for $F(0,T)$ we can solve the nonlinear least squares problem

$$\min_{\alpha,\sigma}\sum_T(\sigma_{F(0,T)}\sqrt{\frac{2\alpha T}{1-e^{-2\alpha T}}}-\sigma)^2$$

to find α and σ.

5.1.8 Natural Gas Market Revisited

We can acomodate the concept of seasonality in the context of Black-Karasinski in a nice way. Once we have solved the optimization problem we fix α and compute the monthly residuals for σ. For a volatility curve with a seasonal pattern this procedure takes care of the time factor and we are left with a curve resembling a sinusoid. We can then extend the "cash volatilities" as far as we want and let mean reversion do its thing. Extending the vol curve for natural gas that we have seen before we obtain:

5.1.9 Black-Karasinski. Conclusion

Even when we have now a more suited model, it is fair to say that we have not done anything fundamentally different from B-S-M. We have created a stochastic process which produces the "right" shape for the vol curve but we have still a fundamental problem. Different futures contracts are highly correlated but that correlation is seldom 1. It can be easily seen that the model we have just presented correlates all futures contracts in a perfect way. To get rid of this drawback some other models have been proposed. One example is the model of Gibson and Schwartz. They solved the problem by adding a second source of randomness due to the **convenience yield**.

6 Term Structure Models: Second Approach

A different approach was used by David Heath, Robert Jarrow and Andy Morton. They moved away from B-S-M by realizing that the goal now is to model a 1-dimensional (random) object. So, they did not worry about the process followed by the cash variable and, instead, started by modelling the futures (forwards in their case since they were working with interest rates) curve. As an extra advantage their approach allows for the introduction of more sources of randomness in a much more natural way.

6.1 Heath-Jarrow-Morton. Cortazar-Schwartz

The translation of H-J-M to the world of commodities was first done by Gonzalo Cortazar and Eduardo Schwartz (1994) in the context of copper futures. In this model, the dynamics of the futures curve is given by

$$\frac{dF(t,T)}{F(t,T)} = \sum_{i=1}^{n} \sigma_i(t,T)dW_i \tag{22}$$

where $F(t,T)$ represents the price at time t of a future with delivery at time T.

Remark: Black-Karasinski can be rewritten in this context as

$$\frac{dF(t,T)}{F(t,T)} = \sigma e^{-\alpha(T-t)}dW. \tag{23}$$

From here we see that the shock at time t is the same acrosss maturities and then the curve can only move in "parallel".

6.2 Principal Components Analysis and HJM

We have stressed before the fact that correlation across contracts does not need to be 1. However, that correlation could turn out to be very high for adjacent contracts. We can extract the main drivers in the dynamics of the curve and build our model based on those. To find the σ_i's we diagonalize the correlation matrix to find the principal components and their corresponding ordering and importance. Robert Litterman and José Scheinkman (1991) found three main factors when studying the dynamics of the (treasury) yield curve. These factors were identified as:

1) Level: The main vector in explaining variances is a parallel shift in the curve.

2) Steepness: Next in importance comes a factor that moves long maturities and short maturities in opposite directions.

3) Curvature: Last, they find a factor that twists the curve.

Litterman-Scheinkman's results were "rediscovered" by Cortazar and Schwartz in their analysis of the copper futures curve. They have also found that level, steepness, and curvature explain a large portion (roughly 99%) of the variance. This is somewhat surprising, it would seem that random curves in the context of fixed income and in the context of commodities follow similar patterns. We will try to give an answer to this "puzzle".

6.3 Example

We have run a principal components analysis to weekly data for the crude oil market for the period 1986-1998. Here are the first 6 eigenvectors.

6.4 Asymptotic Diagonalization

One of the things that seems to be true across different markets is the fact that adjacent contracts (or rates) are highly correlated, and that correlation decays for non-adjacent ones. If we assume that the correlation for adjacent contracts is a constant number ρ then we can prove that the rate of decay non-adjacent ones is ρ^{i-j} where i and j are two different maturities. In other words the correlation matrix looks like:

$$a_{ij} = \rho^{|i-j|} \qquad 1 \leq i,j \leq n$$

i.e.

$$\begin{pmatrix} 1 & \rho & \rho^2 & \dots & \dots & \rho^n \\ \rho & 1 & \rho & \dots & \dots & \rho^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \dots & 1 & \rho \\ \rho^n & \rho^{n-1} & \rho^{n-2} & \dots & \rho & 1 \end{pmatrix}$$

Using this hypotesis we can prove that for ρ sufficiently close to 1 the eigenvectors of the correlation matrix "behave" like cosines.

6.4.1 The Continuous Case.

Based on the previous considerations we study the asymptotic behaviour of the eigenvalues and eigenvectors of the operator:

$$Kf(x) = \int_0^{\pi} \rho^{c|y-x|} f(y)dy \tag{24}$$

$K : L^2(0,\pi) \longrightarrow L^2(0,\pi)$. We find that:

$$K \text{ Cos}(nx) = \frac{c \log(\rho)}{n^2 + c^2 \log(\rho)^2}(\rho^{cx} + \rho^{c(\pi-x)} \text{ Cos}(n\pi) - 2 \text{ Cos}(nx)) \tag{25}$$

As we are interested in the case of ρ close to (but less than) 1, we can, for n odd, write this as:

$$K \text{ Cos}(nx) = \frac{-2c \log(\rho)}{n^2 + c^2 \log(\rho)^2}(\text{ Cos}(nx) + \frac{c\pi - 2cx}{2} \log(\rho) + o(\log(\rho)^2) \tag{26}$$

from where deduce that for n odd

$$\| \text{ Cos}(nx) - \frac{1}{\lambda_n} K \text{ Cos}(nx)\|_2 = \|\frac{c\pi - 2cx}{2} \log(\rho) + o(\log(\rho)^2)\|_2 \overset{\rho \to 1}{\longrightarrow} 0 \tag{27}$$

where

$$\lambda_n = \frac{-2c \log(\rho)}{n^2 + c^2 \log(\rho)^2} \tag{28}$$

To deal with the case in which n is even we apply K to $f(x) = \text{Cos}(nx) + \frac{\lambda_n}{\pi}$. After some computations we get

$$K(\text{Cos}(nx) + \frac{\lambda_n}{\pi}) = \lambda_n(\text{Cos}(nx) + o(\log(\rho))). \tag{29}$$

So, for n even we obtain

$$\|\text{Cos}(nx) + \frac{\lambda_n}{\pi} - \frac{1}{\lambda_n}K(\text{Cos}(nx) + \frac{\lambda_n}{\pi})\|_2 = \|o(\log(\rho))\|_2 \xrightarrow{\rho \to 1} 0. \tag{30}$$

For $n = 0$ we have

$$K1(x) = \frac{-2 + \rho^{cx} + \rho^{c(\pi - x)}}{\rho^c} \xrightarrow{\rho \to 1} \pi \tag{31}$$

Therefore, we can deduce that whenever ρ is roughly constant and close to 1, the principal components will be perturbations of integer frequencies.

7 Conclusions

It is a fact that sophisticated models are needed when pricing derivatives depending on more than one asset. Commodity markets have profited a lot from the bulk of the research coming from the interest rates literature. Yield curves are, in some sense, cleaner than commodity forward curves. They do not follow seasonal patterns, they do not depend on different crops, etc. A lot of work remains to be done in bringing term structure models fully integrated with the commodities derivatives world.

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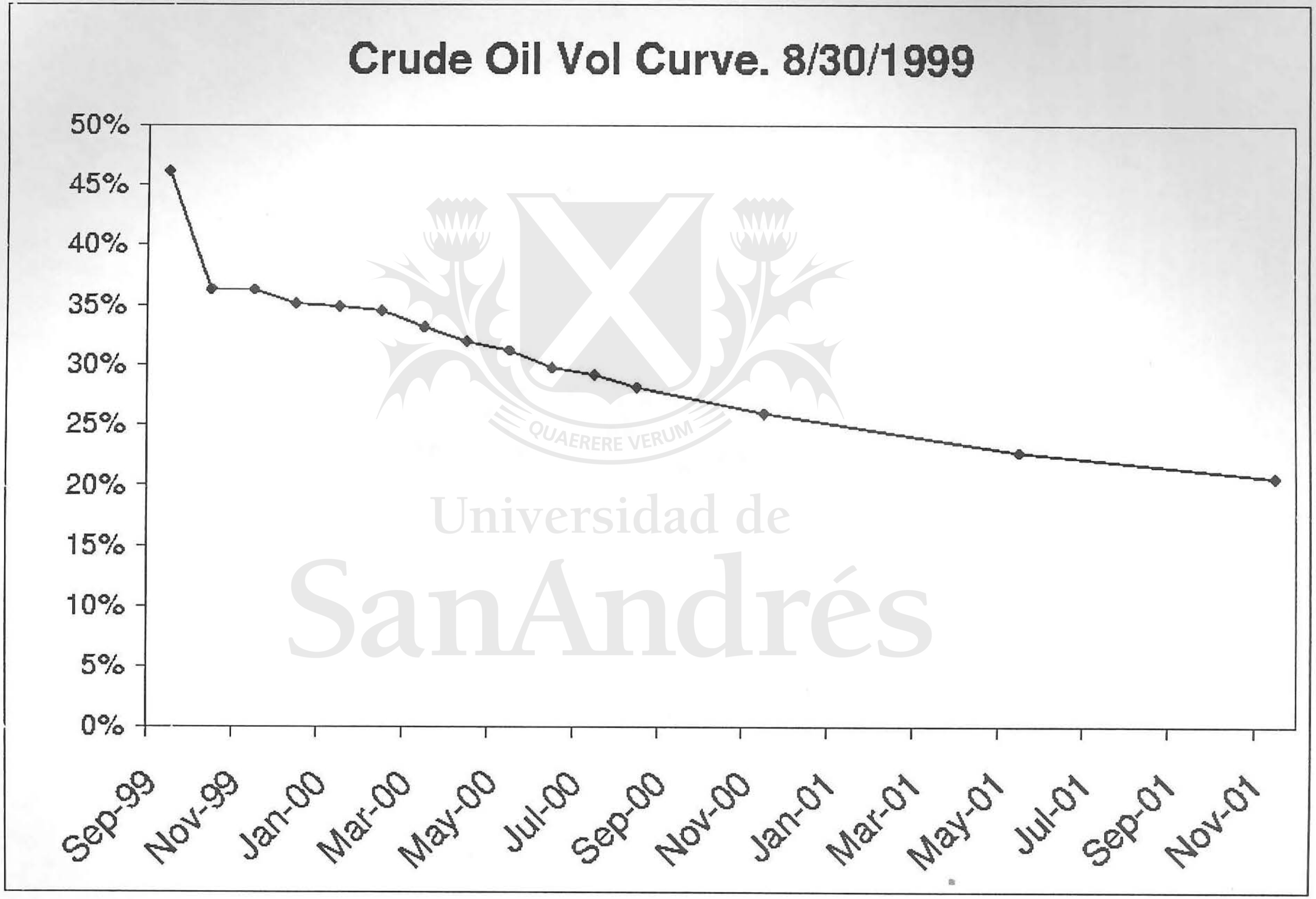
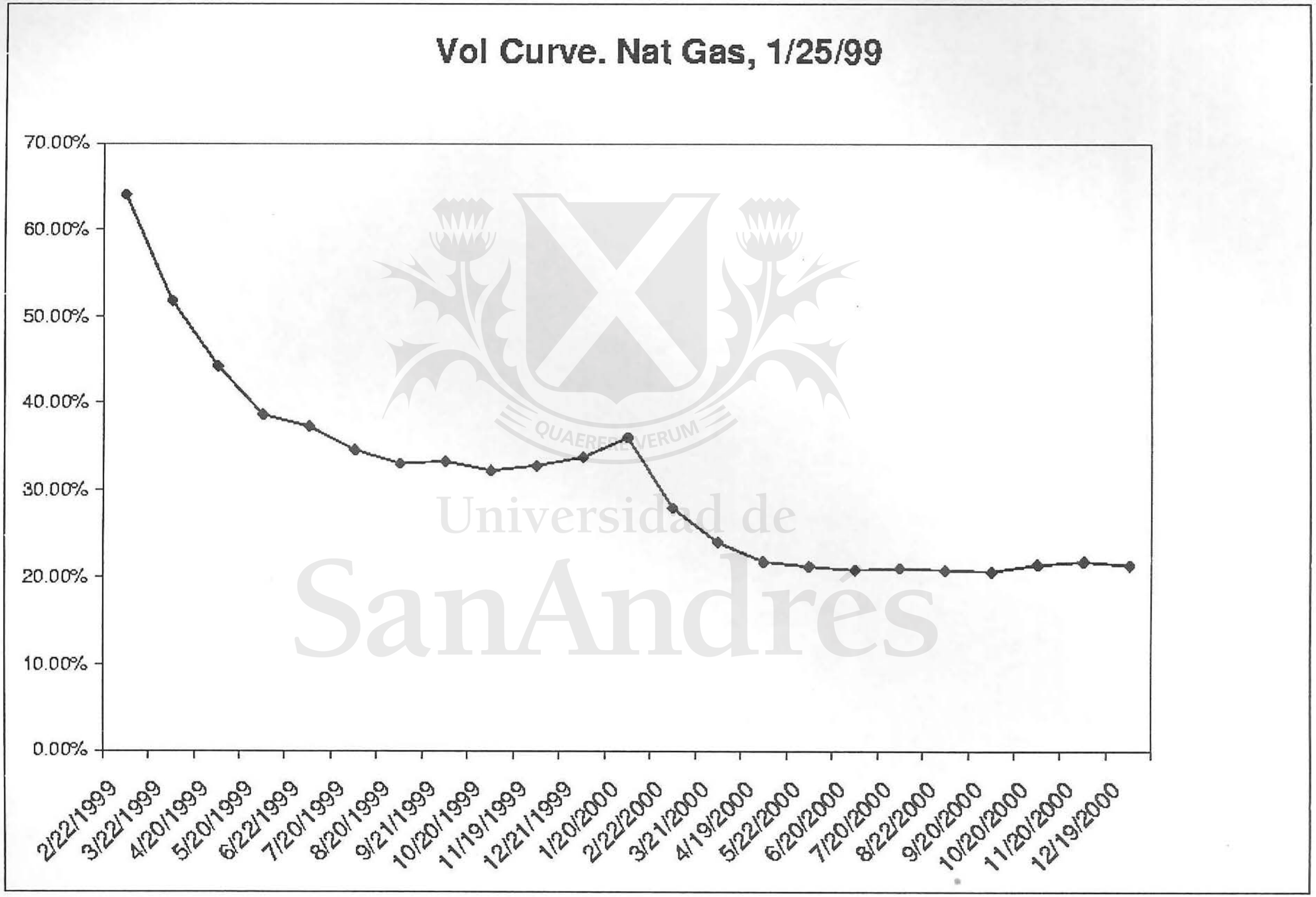

Crude Oil Vol Curve. 8/30/1999
50%
45%
40%
35%
30%
25%
20%
15%
10%
5%
0%
Sep-99
Nov-99
Jan-00
Mar-00
May-00
Jul-00
Sep-00
Nov-00
Jan-01
Mar-01
May-01
Jul-01
Sep-01
Nov-01



Vol Curve. Nat Gas, 1/25/99
70.00%
60.00%
50.00%
40.00%
30.00%
20.00%
10.00%
0.00%
2/22/1999
3/22/1999
4/20/1999
5/20/1999
6/22/1999
7/20/1999
8/20/1999
9/21/1999
10/20/1999
11/19/1999
12/21/1999
1/20/2000
2/22/2000
3/21/2000
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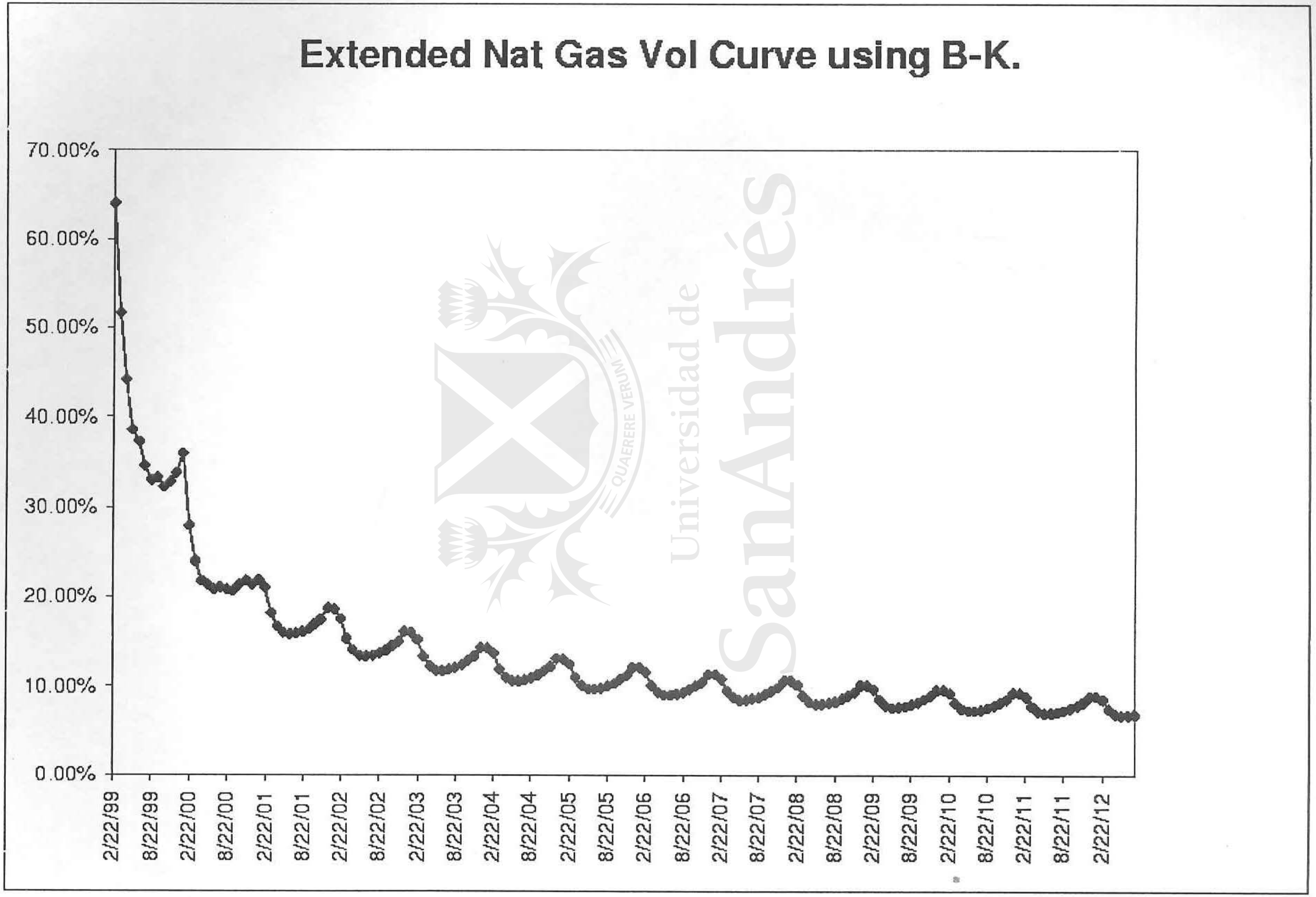
11/20/2000
12/19/2000


Extended Nat Gas Vol Curve using B-K.
70.00%
60.00%
50.00%
40.00%
30.00%
20.00%
10.00%
0.00%
2/22/99
8/22/99
2/22/00
8/22/00
2/22/01
8/22/01
2/22/02
8/22/02
2/22/03
8/22/03
2/22/04
8/22/04
2/22/05
8/22/05
2/22/06
8/22/06
2/22/07
8/22/07
2/22/08
8/22/08
2/22/09
8/22/09
2/22/10
8/22/10
2/22/11
8/22/11
2/22/12