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**“Ideals of Multilinear Forms - a Limit Order  
Approach”**

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# Ideals of multilinear forms - a limit order approach

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## Abstract

A general theory of limit orders for ideals of multilinear forms is developed. We relate the limit order of an ideal to those of its maximal hull and its adjoint ideal. We study the limit orders of the ideals of dominated and multiple summing multilinear forms. Finally, estimates of the diagonal of a (non-necessarily diagonal) multilinear form are presented, in terms of the limit order of the ideals to which it belongs.

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## Introduction

In 1983, A. Pietsch [20] presented his “designs of a theory” for ideals of multilinear forms. His work provided a general framework from which different lines of investigation developed. Some ideals of multilinear forms appeared as the multilinear natural extension of ideals of linear operators (e.g., nuclear and integral multilinear forms). However, it is not always clear what the multilinear analogous of a linear operators ideal should be. For example, the ideal of absolutely  $r$ -summing operators lead to the development of many ideals of multilinear forms: absolutely  $r$ -summing,  $r$ -dominated, multiple  $r$ -summing, etc. Independently of their linear origin, many ideals of multilinear forms were studied by their own interest and also in relation to ideals of polynomials and holomorphy types. In any case, the theory of ideals of multilinear forms allows to deal with all the different situations in a unified way.

In the linear theory, the calculus of limit orders proved a useful tool, specially to compare different ideals of linear operators. The concept of limit order for

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multilinear forms was introduced in [4]. As an application, it could be shown that some properties of bilinear ideals were no longer valid for the  $n$ -linear case ( $n \geq 3$ ).

The first aim of this article is to develop a bit further the general theory of limit orders for multilinear forms. Even though some of our proofs are adaptations of the linear analogous results, we chose to present them here. Our motivation was to give a self-contained treatment of the subject, since the linear versions of these results come from different sources and with different notations. The general theory developed, we focus on some particular ideals of multilinear forms.

In the first section, we recall the definitions of limit order and present some general properties. The second section deals with maximal and adjoint ideals of multilinear forms and their corresponding limit orders. In section 3, the ideal of  $r$ -dominated  $n$ -linear forms is shown to be dual to a tensor norm. This allows us to describe its adjoint ideal. In section 4 we estimate the limit order of the ideal of multiple 1-summing forms. Finally, in section 5 we study the diagonal of a multilinear form. We show that the limit order of a multilinear ideal gives estimates of the diagonal of any multilinear form in the ideal.

Given  $X, Y$  Banach spaces, we denote by  $\mathcal{L}(X; Y)$  the space of continuous linear mappings  $T : X \rightarrow Y$ . If  $X_1, \dots, X_n$  and  $Y$  are Banach spaces,  $\mathcal{L}(X_1, \dots, X_n; Y)$  denotes the space of continuous  $n$ -linear mappings  $T : X_1 \times \dots \times X_n \rightarrow Y$ . Whenever  $X_1 = \dots = X_n = X$  and  $Y = \mathbb{C}$ , the space of continuous  $n$ -linear mappings is simply denoted by  $\mathcal{L}^n(X)$ . We are going to deal with mappings  $T \in \mathcal{L}^n(\ell_p)$ . We denote by  $x_1, \dots, x_n$  the elements in  $\ell_p$ . If  $x$  is a sequence we write  $x = (x(k))_{k=1}^\infty$ , with  $x(k) \in \mathbb{C}$ . Given a sequence  $(x(k))_k \subseteq \mathbb{C}$ , its  $p$ -norm  $(\sum |x(k)|^p)^{1/p}$  will be denoted by  $\ell_p(x(i))$  or  $\|x\|_p$ . If we have two sequences  $(x(k))_k$  and  $(y(k))_k$  we will denote  $(x(k))_k \prec (y(k))_k$  if there exist a constant  $C$  such that  $x(k) \leq C y(k)$  for every  $k$ . We will denote  $(x(k))_k \asymp (y(k))_k$  if  $(x(k))_k \prec (y(k))_k$  and  $(y(k))_k \prec (x(k))_k$ .

Although all the results in the article are proved for complex Banach spaces, standard modifications can be made to obtain the real version of most of them.

## 1 Limit order of ideals of multilinear forms

In [4], the limit order for ideals of multilinear forms is defined. We recall the definitions and some basic facts.

Given a sequence  $\sigma$ , the associated diagonal operator  $D_\sigma$  from  $\ell_p$  to  $\ell_q$  is defined by  $D_\sigma(x) = (\sigma(k)x(k))_k$ . With this, given an operator ideal  $\mathfrak{A}$ , the limit order  $\lambda(\mathfrak{A}; p, q)$  (see [19, Section 14.4]) is the infimum over all  $\lambda \geq 0$  such that every diagonal operator  $D_\sigma : \ell_p \rightarrow \ell_q$  with  $\sigma \in \ell_{1/\lambda}$  belongs to  $\mathfrak{A}(\ell_p, \ell_q)$ .

Ideals of multilinear forms were introduced in [20]. Let us recall the definition. An **ideal of multilinear forms**  $\mathfrak{A}$  is a subclass of  $\mathcal{L}$ , the class of all multilinear forms such that, for any Banach spaces  $X_1, \dots, X_n$  the set

$$\mathfrak{A}(X_1, \dots, X_n) = \mathfrak{A} \cap \mathcal{L}(X_1, \dots, X_n)$$

satisfies

1. For any  $\gamma_1 \in X'_1, \dots, \gamma_n \in X'_n$ , the mapping  $(x_1, \dots, x_n) \mapsto \gamma_1(x_1) \cdots \gamma_n(x_n)$  belongs to  $\mathfrak{A}(X_1, \dots, X_n)$ .
2. If  $S, T \in \mathfrak{A}(X_1, \dots, X_n)$ , then  $S + T \in \mathfrak{A}(X_1, \dots, X_n)$ .
3. If  $T \in \mathfrak{A}(X_1, \dots, X_n)$  and  $S_i \in \mathcal{L}(Y_i, X_i)$  for  $i = 1, \dots, n$ , then  $T \circ (S_1, \dots, S_n) \in \mathfrak{A}(Y_1, \dots, Y_n)$ .

An ideal of multilinear forms is called **(quasi-) normed** if for each  $X_1, \dots, X_n$  there is a (quasi-) norm  $\|\cdot\|_{\mathfrak{A}(X_1, \dots, X_n)}$  in  $\mathfrak{A}(X_1, \dots, X_n)$  such that

1.  $\|(x_1, \dots, x_n) \mapsto \gamma_1(x_1) \cdots \gamma_n(x_n)\|_{\mathfrak{A}(X_1, \dots, X_n)} = \|\gamma_1\| \cdots \|\gamma_n\|$ .
2.  $\|T \circ (S_1, \dots, S_n)\|_{\mathfrak{A}(Y_1, \dots, Y_n)} \leq \|T\|_{\mathfrak{A}(X_1, \dots, X_n)} \cdot \|S_1\| \cdots \|S_n\|$ .

Following the spirit of the definition of limit order for operator ideals in [4], the concept of limit order was defined for ideals of multilinear forms. First of all, if  $T \in \mathcal{L}(^n \ell_p)$ , we call it **diagonal** if there exists a sequence  $\alpha = (\alpha(k))_k$  such that for all  $x_1, \dots, x_n \in \ell_p$  we can write

$$T(x_1, \dots, x_n) = \sum_k \alpha(k) x_1(k) \cdots x_n(k).$$

We denote by  $T_\alpha$  the diagonal multilinear mapping given by the sequence  $\alpha$ . With this the limit order can be defined.

**Definition 1.1.** *Let  $\mathfrak{A}$  be an ideal of multilinear forms. For  $1 \leq p \leq \infty$ , the limit order  $\lambda_n(\mathfrak{A}; p)$  is given by:*

$$\lambda_n(\mathfrak{A}; p) = \inf\{\lambda : \text{for each } \alpha \in \ell_{1/\lambda}, T_\alpha \text{ belongs to } \mathfrak{A}(^n \ell_p)\}$$

From the definition we have that  $0 \leq \lambda_n(\mathfrak{A}; p) \leq 1$  for every ideal  $\mathfrak{A}$  and all  $p$  and  $n$ . Also, if  $\mathfrak{A}, \mathfrak{B}$  are ideals such that  $\mathfrak{A} \subseteq \mathfrak{B}$ , then  $\lambda_n(\mathfrak{A}; p) \geq \lambda_n(\mathfrak{B}; p)$ .

With almost the same proof as in [19, Section 14.4], we obtain alternative expressions for  $\lambda_n(\mathfrak{A}; p)$ . First, we have:

$$\lambda_n(\mathfrak{A}; p) = \inf\{\lambda : \text{if } \alpha = (k^{-\lambda})_k, \text{ then } T_\alpha \text{ belongs to } \mathfrak{A}(^n \ell_p)\}.$$

Given  $N \in \mathbb{N}$ , we define the  $n$ -linear form  $\Phi_N$  on  $\mathbb{C}^N$  by:

$$\Phi_N(x_1, \dots, x_n) = \sum_{k=1}^N x_1(k) \cdots x_n(k).$$

With this, if  $\mathfrak{A}$  is quasi-normed and complete, then  $\lambda_n(\mathfrak{A}; p)$  is the infimum of all  $\lambda \geq 0$  such that

$$\|\Phi_N\|_{\mathfrak{A}(^n \ell_p^N)} \leq CN^\lambda \tag{1}$$

for all  $N \in \mathbb{N}$ , where  $C > 0$  is a constant.

Clearly, the definition of limit order can be generalized to  $n$ -linear forms on  $\ell_{p_1} \times \cdots \times \ell_{p_n}$ . All the results in this article can be extended to this situation. However, for the reader's convenience, we prefer to state them in the simple case  $p_1 = \cdots = p_n = p$ .

It is also possible to define the limit order of ideals of  $n$ -homogeneous polynomials. But this will not lead us to new horizons. Indeed, in [12] the authors showed that, given a  $\lambda$ -normed ideal of  $n$ -homogeneous polynomials  $\mathcal{Q}$ , there exists a  $\lambda$ -normed ideal of  $n$ -linear forms  $\mathcal{Q}^\vee$  with the following property:

A polynomial  $P$  is in  $\mathcal{Q}$  if and only if its associated symmetric  $n$ -linear form  $\tilde{P}$  is in  $\mathcal{Q}^\vee$ .

Moreover,  $P \leftrightarrow \tilde{P}$  is a one-to-one correspondence between diagonal  $n$ -homogeneous polynomials and diagonal symmetric  $n$ -linear forms. Then, the limit orders of ideals of homogeneous polynomials can be seen as limit orders of ideals of multilinear forms.

Given a diagonal multilinear form  $T_\alpha \in \mathcal{L}({}^n\ell_p)$ , we consider a sequence  $\sigma$  such that  $\sigma(k)^n = \alpha(k)$  for all  $k$ . We take the diagonal operator  $D_\sigma : \ell_p \rightarrow \ell_n$  associated to  $\sigma$  and define a mapping  $\Phi : \ell_n \times \cdots \times \ell_n \rightarrow \mathbb{C}$  by  $\Phi(x_1, \dots, x_n) = \sum_k x_1(k) \cdots x_n(k)$ . The fact that  $T$  is well defined on  $\ell_p$  guarantees that  $D_\sigma(\ell_p) \subset \ell_n$ . Now, the diagonal  $n$ -linear mapping  $T$  can be rewritten as

$$T_\alpha(x_1, \dots, x_n) = \Phi(D_\sigma(x_1), \dots, D_\sigma(x_n)). \quad (2)$$

This decomposition allows to sometimes get some relations between limit orders of ideals of multilinear forms and of ideals of operators.

**Examples:** In [4] the limit orders of nuclear, integral and  $r$ -dominated multilinear forms (see [8] for definitions) are computed:

$$\lambda_n(\mathcal{L}; p) = \begin{cases} 0 & \text{if } p \leq n \\ 1 - \frac{n}{p} & \text{if } p > n \end{cases}$$

$$\lambda_n(\mathcal{N}; p) = \lambda_n(\mathcal{I}; p) = \begin{cases} \frac{n}{p'} & \text{if } 1 \leq p < n' \\ 1 & \text{if } n' \leq p \end{cases}$$

$$\lambda_n(\mathcal{D}_r; p) = n \lambda(\Pi_r; p, n)$$

where  $\lambda(\Pi_r; p, n)$  is the limit order of the ideal of absolutely  $r$ -summing linear operators (see [19, Section 22.4]). The relation of the limit order of  $r$ -dominated  $n$ -linear forms and absolutely  $r$ -summing operators follows from the following result:

**Proposition 1.2.** [4] *Let  $T_\alpha \in \mathcal{L}({}^n\ell_p)$  be diagonal and  $D_\sigma$  its associated diagonal operator. Then  $T_\alpha$  is  $r$ -dominated if and only if  $D_\sigma$  is absolutely  $r$ -summing.*

Now we state some general properties of limit orders. As in the linear case, the following property holds for composition ideals:

**Proposition 1.3.** *Let  $\mathfrak{A}$  be an ideal of  $n$ -linear forms and  $\mathfrak{B}_1, \dots, \mathfrak{B}_n$  linear operator ideals. Then:*

$$\lambda_n(\mathfrak{A} \circ (\mathfrak{B}_1, \dots, \mathfrak{B}_n); p) \leq \lambda(\mathfrak{B}_1; p, s) + \dots + \lambda(\mathfrak{B}_n; p, s) + \lambda_n(\mathfrak{A}; s)$$

As a consequence, since  $\mathfrak{A} = \mathfrak{A} \circ (\mathcal{L}, \dots, \mathcal{L})$ , we have

$$|\lambda_n(\mathfrak{A}; p) - \lambda_n(\mathfrak{A}; p_0)| \leq n \left| \frac{1}{p} - \frac{1}{p_0} \right|.$$

Therefore,  $\lambda_n(\mathfrak{A}; p)$  is a continuous function of  $\frac{1}{p}$ .

## 2 Maximal and adjoint ideals of multilinear mappings

Let  $\mathfrak{A}$  be a quasi-normed ideal of  $n$ -linear forms. The maximal hull  $\mathfrak{A}^{max}$  of  $\mathfrak{A}$  is defined as the class of all  $n$ -linear forms  $T$  such that

$$\|T\|_{\mathfrak{A}^{max}(E_1, \dots, E_n)} := \sup\{\|T\|_{M_1 \times \dots \times M_n} \|_{\mathfrak{A}(M_1, \dots, M_n)} : M_i \subset E_i, \dim M_i < \infty\}$$

is finite.

$\mathfrak{A}^{max}$  is always complete and it is the largest ideal whose quasi-norm coincides with  $\|\cdot\|_{\mathfrak{A}}$  in finite dimensional spaces. Hence, if  $\mathfrak{A}$  is complete, equation (1) says that

$$\lambda_n(\mathfrak{A}; p) = \lambda_n(\mathfrak{A}^{max}; p)$$

for all  $p$ .

A quasi-normed ideal  $\mathfrak{A}$  is called maximal if  $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}}) = (\mathfrak{A}^{max}, \|\cdot\|_{\mathfrak{A}^{max}})$ .

Following [5, 10, 13] we define now the adjoint of an ideal of multilinear mappings. Let  $\mathfrak{A}$  be a normed ideal of  $n$ -linear mappings. If  $M_1, \dots, M_n$  are finite dimensional normed spaces, the multilinear norm  $\|\cdot\|_{\mathfrak{A}}$  induces a tensor norm  $\alpha$  in FIN (the class of all finite dimensional normed spaces) by means of the identification:

$$\left( \bigotimes_{i=1}^n M_i; \alpha \right) \stackrel{1}{=} \mathfrak{A}(M'_1, \dots, M'_n)$$

This norm can be extended to a finitely generated tensor norm on the class of normed spaces by

$$\|s\|_{(\bigotimes_{i=1}^n E_i; \alpha)} := \inf \left\{ \|s\|_{(\bigotimes_{i=1}^n M_i; \alpha)} : M_i \in \text{FIN}(E_i), s \in \bigotimes_{i=1}^n M_i \right\}$$

In this case, the tensor norm  $\alpha$  and the ideal  $\mathfrak{A}$  are said to be associated.

Given a normed ideal  $\mathfrak{A}$  associated to the finitely generated tensor norm  $\alpha$ , its adjoint ideal  $\mathfrak{A}^*$  is defined by

$$\mathfrak{A}^*(E_1, \dots, E_n) := \left( \bigotimes_{i=1}^n E_i; \alpha \right)'.$$

The adjoint ideal is called dual ideal in [10].

The tensor norm associated to  $\mathfrak{A}^*$  is denoted by  $\alpha^*$ . We also have the representation theorem [13, Section 3.2] (see also [10, Section 4]):

$$\mathfrak{A}^{max}(E_1, \dots, E_n) = \left( \bigotimes_{i=1}^n E_i; \alpha^* \right)'.$$

In particular, this shows that the adjoint ideal  $\mathfrak{A}^*$  is maximal.

The limit orders of an ideal and its adjoint ideal are related, as in the linear case, by the following equality (see [5, Sect. 17.19])

**Lemma 2.1.**  $\|\Phi_N\|_{\mathfrak{A}(n\ell_p^N)} \|\Phi_N\|_{\mathfrak{A}^*(n\ell_{p'}^N)} = N$

*Proof.* Let us note first that

$$\begin{aligned} N &= \sum_{k=1}^N \Phi_N(e_k, \dots, e_k) = \Phi_N \left( \sum_{k=1}^N e_k \otimes \dots \otimes e_k \right) \\ &\leq \|\Phi_N\|_{\mathfrak{A}(n\ell_p^N)} \left\| \sum_{k=1}^N e_k \otimes \dots \otimes e_k \right\|_{(\bigotimes_{i=1}^n \ell_p^N; \alpha^*)} = \|\Phi_N\|_{\mathfrak{A}(n\ell_p^N)} \cdot \|\Phi_N\|_{\mathfrak{A}^*(n\ell_{p'}^N)} \end{aligned}$$

For the reverse inequality, let us choose a norm one  $s \in (\bigotimes_{i=1}^n \ell_p^N; \alpha^*)$  such that  $\|\Phi_N\|_{\mathfrak{A}(n\ell_p^N)} = |\Phi_N(s)|$ . Then

$$\|\Phi_N\|_{\mathfrak{A}(n\ell_p^N)} \cdot \|\Phi_N\|_{\mathfrak{A}^*(n\ell_{p'}^N)} = \|\Phi_N(s) \cdot \Phi_N\|_{\mathfrak{A}^*(n\ell_{p'}^N)}.$$

We write  $s = \sum_{i_1, \dots, i_n=1}^N \alpha_{i_1, \dots, i_n} e_{i_1} \otimes \dots \otimes e_{i_n}$  and define  $s_0 := \sum_{i=1}^N \alpha_{i, \dots, i} e_i \otimes \dots \otimes e_i$  the diagonal of  $s$ . Clearly,  $\Phi_N(s) = \Phi_N(s_0)$ .

Now we define  $S : \ell_{p'}^N \rightarrow \ell_{p'}^N$  by  $S(x_1, x_2, \dots, x_N) = (x_2, \dots, x_N, x_1)$ . It is easy to see that for each  $1 \leq j \leq n$ ,  $\sum_{k=1}^N (e_j \otimes \dots \otimes e_j) \circ (S^k, \dots, S^k) = \Phi_N \in \mathfrak{A}^*(n\ell_{p'}^N)$ . Therefore,

$$\begin{aligned} \|\Phi_N(s) \cdot \Phi_N\|_{\mathfrak{A}^*(n\ell_{p'}^N)} &= \|\Phi_N(s_0) \cdot \Phi_N\|_{\mathfrak{A}^*(n\ell_{p'}^N)} \\ &= \left\| \sum_{i=1}^N \alpha_{i, \dots, i} \cdot \sum_{k=1}^N (e_i \otimes \dots \otimes e_i) \circ (S^k, \dots, S^k) \right\|_{\mathfrak{A}^*(n\ell_{p'}^N)} \\ &= \left\| \sum_{k=1}^N s_0 \circ (S^k, \dots, S^k) \right\|_{\mathfrak{A}^*(n\ell_{p'}^N)} \leq N \cdot \|s_0\|_{\mathfrak{A}^*(n\ell_{p'}^N)}. \end{aligned}$$

Here we identify tensors with multilinear forms. Proceeding as in Proposition 5.1 and noting that in this situation the operator  $\Lambda_t$  has norm one, we can see that  $\|s_0\|_{\mathfrak{A}^*({}^n\ell_p^N)} \leq \|s\|_{\mathfrak{A}^*({}^n\ell_p^N)} = \|s\|_{(\otimes^n \ell_p^N; \alpha^*)} = 1$ . This completes the proof.  $\square$

Immediately from this and (1) we have

**Corollary 2.2.** *Let  $\mathfrak{A}$  be a normed ideal of multilinear forms. Then  $\lambda_n(\mathfrak{A}, p) + \lambda_n(\mathfrak{A}^*; p') \geq 1$ .*

The following definition and results are multilinear versions of those introduced in [14] for linear operators. As in [14], we denote  $\ell_n(\mathfrak{A}, p) := \{\alpha \in \ell_\infty : T_\alpha \in \mathfrak{A}({}^n\ell_p)\}$ . The sequence space  $\ell_n(\mathfrak{A}, p)$  is a Banach space if we consider the norm

$$\|\alpha\|_{\ell_n(\mathfrak{A}, p)} = \|T_\alpha\|_{\mathfrak{A}({}^n\ell_p)}$$

**Definition 2.3.** *Let  $\mathfrak{A}$  be an ideal of  $n$ -linear forms and  $1 \leq p \leq \infty$ . We define the **defect** by*

$$d_n(\mathfrak{A}, p) = \inf \left\{ \frac{1}{r} - \frac{1}{s} : \ell_r \subset \ell_n(\mathfrak{A}, p) \subset \ell_s \right\}.$$

**Proposition 2.4.** *Let  $\mathfrak{A}$  be a Banach ideal of  $n$ -linear forms. Then  $d_n(\mathfrak{A}, p)$  is the infimum of  $\lambda - \mu$  where  $\lambda, \mu \geq 0$  are such that*

$$CN^\mu \leq \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)} \leq DN^\lambda \quad (3)$$

for all  $N \in \mathbb{N}$ , for some constants  $C, D > 0$ .

*Proof.* Take  $r, s$  such that  $\ell_r \subset \ell_n(\mathfrak{A}, p) \subset \ell_s$ . From the closed graph theorem, the inclusions  $\ell_r \hookrightarrow \ell_n(\mathfrak{A}, p)$  and  $\ell_n(\mathfrak{A}, p) \hookrightarrow \ell_s$  are continuous. Thus, there exist constants  $C$  and  $D$  such that, for all  $N \in \mathbb{N}$ ,

$$CN^{\frac{1}{s}} \leq \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)} = \|(1, \dots, 1, 0, \dots)\|_{\ell_n(\mathfrak{A}, p)} \leq DN^{\frac{1}{r}}.$$

Conversely, suppose  $CN^{\frac{1}{s}} \leq \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)} \leq DN^{\frac{1}{r}}$  for all  $N \in \mathbb{N}$ . It is enough to show that, for all  $\varepsilon > 0$ , we have  $\ell_{r-\varepsilon} \subset \ell_n(\mathfrak{A}, p) \subset \ell_{s+\varepsilon}$ .

The first inclusion follows from the equivalence between the definitions of limit order. For the second inclusion, let  $\sigma \in \ell_n(\mathfrak{A}, p)$ . We first consider the case in which  $(\sigma(k))_k$  converges to 0. We can assume that  $(|\sigma(k)|)_k$  is not increasing. We can factor  $\Phi_N$  as

$$\begin{array}{ccccccc} \ell_p^N & \times & \dots & \times & \ell_p^N & \xrightarrow{\Phi_N} & \mathbb{C} \\ D_\alpha \downarrow & & & & D_\alpha \downarrow & \nearrow & \\ \ell_p^N & \times & \dots & \times & \ell_p^N & & T_\sigma \end{array}$$

where  $\alpha = \sigma^{-1/n}$ . Therefore,

$$CN^{\frac{1}{s}} \leq \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)} \leq \|T_\sigma\|_{\mathfrak{A}({}^n\ell_p^N)} \cdot \|D_\alpha\|^n \leq \|T_\sigma\|_{\mathfrak{A}({}^n\ell_p)} \cdot |\sigma(N)|^{-1}$$



Consequently,  $|\sigma_N| \leq \frac{\|T_\sigma\|_{\mathfrak{A}(n\ell_p)}}{DN^{1/s}}$  for all  $N \in \mathbb{N}$ . This implies that  $\sigma \in \ell_{s+\varepsilon}$ .

Now we assume that  $(\sigma(k))_k$  does not converge to 0. Let us take constants  $a$  and  $b$  and a subsequence of  $(\sigma(k))_k$  such that  $a \leq \sigma(k_i) \leq b$  for all  $i$ . We will see that in this case, both  $\ell_n(\mathfrak{A}, p)$  and  $\ell_s$  coincide with  $\ell_\infty$ .

Let  $\beta \in \ell_\infty$ . We define  $S : \ell_p \rightarrow \ell_p$  by

$$S(e_i) = \left( \frac{\beta(i)}{\sigma(k_i)} \right)^{\frac{1}{n}} e_{k_i}$$

Since  $T_\beta = T_\sigma \circ (S, \dots, S)$ , we have that  $T_\beta \in \mathfrak{A}(n\ell_p)$  and then  $\beta \in \ell_n(\mathfrak{A}, p)$ .

Now, since  $\ell_n(\mathfrak{A}, p) = \ell_\infty$ , we have  $\Phi = T_{(1,1,\dots)} \in \mathfrak{A}(n\ell_p)$ . From the inequality  $CN^{\frac{1}{s}} \leq \|\Phi_N\|_{\mathfrak{A}(n\ell_p^N)} \leq \|\Phi\|_{\mathfrak{A}(n\ell_p)}$  for all  $N$ , we obtain  $s = \infty$ .  $\square$

**Proposition 2.5.** *Let  $\mathfrak{A}$  be a Banach ideal of  $n$ -linear forms and  $\mathfrak{A}^*$  its adjoint ideal. Then*

$$\lambda_n(\mathfrak{A}; p) + \lambda_n(\mathfrak{A}^*; p') = 1 + d_n(\mathfrak{A}; p).$$

*Proof.*

$$\begin{aligned} \lambda_n(\mathfrak{A}; p) + \lambda_n(\mathfrak{A}^*; p') &= \inf\{\lambda > 0 : \exists D > 0 \text{ with } \|\Phi_N\|_{\mathfrak{A}(n\ell_p^N)} \leq DN^\lambda, \forall N\} \\ &\quad + \inf\{\nu > 0 : \exists C > 0 \text{ with } \|\Phi_N\|_{\mathfrak{A}^*(n\ell_p^N)} \leq CN^\nu, \forall N\} \\ &= \inf\{\lambda > 0 : \exists D > 0 \text{ with } \|\Phi_N\|_{\mathfrak{A}(n\ell_p^N)} \leq DN^\lambda, \forall N\} \\ &\quad + \inf\{\nu > 0 : \exists \tilde{C} > 0 \text{ with } \tilde{C}N^{1-\nu} \leq \|\Phi_N\|_{\mathfrak{A}(n\ell_p^N)}, \forall N\} \\ &= 1 + \inf\{\lambda - \mu : \tilde{C}N^\mu \leq \|\Phi_N\|_{\mathfrak{A}(n\ell_p^N)} \leq DN^\lambda\} \\ &= 1 + d_n(\mathfrak{A}; p). \end{aligned}$$

$\square$

Note that Corollary 2.2 can also be obtained as a consequence of Proposition 2.5. We also get the following analogous to [14, Corollary 1].

**Corollary 2.6.** *Let  $\mathfrak{A}$  be a Banach ideal. The following are equivalent:*

- (a)  $\lambda_n(\mathfrak{A}; p) + \lambda_n(\mathfrak{A}^*; p') = 1$ .
- (b) *There exists  $r > 0$  such that for all  $\varepsilon > 0$ ,  $\ell_{r-\varepsilon} \subset \ell_n(\mathfrak{A}, p) \subset \ell_{r+\varepsilon}$ .*
- (c) *There exists  $\lambda \geq 0$  such that for all  $\varepsilon > 0$  and all  $N \in \mathbb{N}$ ,  $CN^{\lambda-\varepsilon} \leq \|\Phi_N\|_{\mathfrak{A}(n\ell_p^N)} \leq DN^{\lambda+\varepsilon}$  for some constants  $C, D > 0$ .*
- (d)  $\left( \frac{\log \|\Phi_N\|_{\mathfrak{A}(n\ell_p^N)}}{\log N} \right)_{N \in \mathbb{N}}$  *converges as  $N \rightarrow \infty$ .*

Moreover, if these equivalences hold, then  $\frac{1}{r} = \lambda = \lambda_n(\mathfrak{A}, p)$

**Remark 2.7.** The definition of limit order implies that, for all  $\varepsilon > 0$ ,

$$\ell_{1/\lambda_n(\mathfrak{A}, p) - \varepsilon} \subset \ell_n(\mathfrak{A}, p).$$

Therefore, the equality  $\lambda_n(\mathfrak{A}; p) + \lambda_n(\mathfrak{A}^*; p') = 1$  is equivalent to the inclusion  $\ell_n(\mathfrak{A}, p) \subset \ell_{1/\lambda_n(\mathfrak{A}, p) + \varepsilon}$  for all  $\varepsilon > 0$ .

### 3 Dominated multilinear forms

Absolutely summing and  $p$ -summing operators (see [5, Ch. 11] or [19, Ch. 17]) admit different generalizations to the multilinear setting, depending on the different properties that we want to preserve. One of these possible generalizations is done attending to the Pietsch's Domination Theorem [19, Sect. 17.3.2]. With this we obtain the  $r$ -dominated multilinear mappings. In [4] the limit order of  $r$ -dominated  $n$ -linear forms was computed. Our aim in this section is to relate the adjoint ideal of  $r$ -dominated forms with the ideal of  $r$ -integral forms (see definitions below) and use the results in [4] and in the previous sections to compute the limit order of the latter. Let us recall first some definitions.

A sequence in a Banach space is **weakly  $p$ -summable** if  $(\gamma(x_n))_n \in \ell_p$  for all  $\gamma \in X'$ . The space of weakly  $p$ -summable sequences endowed with the norm

$$w_p((x_n)_n) = \sup_{\gamma \in B_{X'}} \left( \sum_n |\gamma(x_n)|^p \right)^{1/p}$$

is a Banach space.

A map  $T \in \mathcal{L}(X_1, \dots, X_n; Y)$  is said to be **absolutely  $(s; r_1, \dots, r_n)$ -summing** (where  $\frac{1}{s} \leq \frac{1}{r_1} + \dots + \frac{1}{r_n}$ ) [1, 15] if there exists  $C > 0$  such that for any finite choice of elements  $x_j^i \in X_j$ ,  $j = 1, \dots, n$ ,  $i = 1, \dots, m$  we have

$$\left( \sum_{i=1}^m \|T(x_1^i, \dots, x_n^i)\|^s \right)^{1/s} \leq C \cdot w_{r_1}(x_1^i) \cdots w_{r_n}(x_n^i).$$

A map  $T \in \mathcal{L}((X_1, \dots, X_n; Y)$  is said to be  **$r$ -dominated** [21, 17] if it is absolutely  $(r/n; r, \dots, r)$ -summing; that is, there exists  $C > 0$  such that for every  $x_j^i \in X_j$ ,  $j = 1, \dots, n$ ,  $i = 1, \dots, m$ ,

$$\left( \sum_{i=1}^m \|T(x_1^i, \dots, x_n^i)\|^{r/n} \right)^{n/r} \leq C \cdot w_r(x_1^i) \cdots w_r(x_n^i).$$

We denote by  $\mathcal{D}_r(^n X)$  the space of  $r$ -dominated  $n$ -linear forms on  $X$ .

First we show that  $r$ -dominated  $n$ -linear forms are dual to a tensor norm whenever  $r \geq n$ . Next, this duality will be used to study the adjoint ideal  $\mathcal{D}_r^*$ .

For  $r \geq n$ , we define in  $\otimes_{i=1}^n X_i$ ,

$$\alpha_{r'}^n(s) = \inf \left\{ \ell_u(\lambda(i)) \cdot w_r(x_i^1) \cdots w_r(x_i^n) : s = \sum_{i=1}^N \lambda(i) \cdot x_i^1 \otimes \cdots \otimes x_i^n \right\}$$

where  $\frac{1}{u} + \frac{n}{r} = 1$ .

A straightforward application of [9, §1.2] gives

**Proposition 3.1.**  $\alpha_{r'}^n$  is a finitely generated tensor norm of order  $n$ .

Now we can show the desired duality:

**Proposition 3.2.** *If  $r \geq n$ ,  $\mathcal{D}_r({}^n X) = (\bigotimes^n X; \alpha_{r'}^n)'$ .*

*Proof.* Let  $T \in \mathcal{D}_r({}^n X)$ . For  $s = \sum_{i=1}^N \lambda(i) \cdot x_i^1 \otimes \cdots \otimes x_i^n \in \bigotimes^n X$  we have

$$\begin{aligned} |T(s)| &= \left| \sum_{i=1}^N \lambda(i) \cdot T(x_i^1, \dots, x_i^n) \right| \leq \ell_{\frac{r}{n}}(|T(x_i^1, \dots, x_i^n)|) \cdot \ell_u(\lambda(i)) \\ &\leq \|T\|_{\mathcal{D}_r({}^n X)} \cdot w_r(x_i^1) \cdots w_r(x_i^n) \cdot \ell_u(\lambda(i)). \end{aligned}$$

Since this is valid for any representation of  $s$ , we obtain that  $T \in (\bigotimes^n X; \alpha_{r'}^n)'$  and  $\|T\|_{(\bigotimes^n X; \alpha_{r'}^n)'} \leq \|T\|_{\mathcal{D}_r({}^n X)}$ .

Conversely, let  $T \in (\bigotimes^n X; \alpha_{r'}^n)'$ . For any sequences  $(x_i^1)_{i=1}^N, \dots, (x_i^n)_{i=1}^N$  in  $X$ , there exist scalars  $\lambda(1), \dots, \lambda(N)$  with  $\ell_u(\lambda(i)) = 1$  such that

$$\begin{aligned} \ell_{\frac{r}{n}}(|T(x_i^1, \dots, x_i^n)|) &= \sum_{i=1}^N \lambda(i) \cdot T(x_i^1, \dots, x_i^n) \\ &= T\left(\sum_{i=1}^N \lambda(i) \cdot x_i^1 \otimes \cdots \otimes x_i^n\right) \\ &\leq \|T\|_{(\bigotimes^n X; \alpha_{r'}^n)'} \cdot \alpha_{r'}^n\left(\sum_{i=1}^N \lambda(i) \cdot x_i^1 \otimes \cdots \otimes x_i^n\right) \\ &\leq \|T\|_{(\bigotimes^n X; \alpha_{r'}^n)'} \cdot w_r(x_i^1) \cdots w_r(x_i^n) \cdot \ell_u(\lambda(i)). \end{aligned}$$

Thus,  $T \in \mathcal{D}_r({}^n X)$  and  $\|T\|_{\mathcal{D}_r({}^n X)} \leq \|T\|_{(\bigotimes^n X; \alpha_{r'}^n)'}$ .  $\square$

Inspired by [5, Chapters 17 and 18], we study  $\mathcal{D}_r^*$ , the adjoint ideal to the ideal of  $r$ -dominated multilinear mappings.

Note that, since  $\mathcal{D}_r$  is a maximal ideal, we have  $\mathcal{D}_r = (\mathcal{D}_r^*)^*$ . Therefore, for  $M_1, \dots, M_n \in \text{FIN}$ ,

$$\mathcal{D}_r^*(M_1, \dots, M_n) \stackrel{1}{=} \left( \bigotimes_{i=1}^n M_i; \alpha_{r'}^n \right).$$

Let  $T \in \mathcal{D}_r^*(M_1, \dots, M_n)$  and fix  $\varepsilon > 0$ .  $T$  admits a representation in the following way:

$$T(x_1, \dots, x_n) = \sum_{k=1}^N \lambda(k) \cdot \gamma_k^1(x_1) \cdots \gamma_k^n(x_n), \quad (4)$$

where  $(\lambda(k))_k \subset \mathbb{C}$ ,  $(\gamma_k^i)_k \subset M_i'$  satisfy

$$\ell_u(\lambda(k)) \cdot w_r(\gamma_k^1) \cdots w_r(\gamma_k^n) = \|T\|_{\mathcal{D}_r^*(M_1, \dots, M_n)} \cdot (1 + \varepsilon)$$

with  $\frac{1}{u} + \frac{n}{r} = 1$ .

Then we can factor  $T$  as:

$$\begin{array}{ccc} M_1 & \times \cdots \times & M_n & \xrightarrow{T} & \mathbb{C} \\ R^1 \downarrow & & R^n \downarrow & \nearrow & \\ \ell_r^N & \times \cdots \times & \ell_r^N & & T_\lambda \end{array} \quad (5)$$

where  $R^i(x) = (\gamma_k^i(x))_{k=1}^N$ . Since  $\|T_\lambda\| = \ell_u(\lambda(k))$  and  $\|R^i\| = w_r(\gamma_k^i)$ , we have

$$\|T_\lambda\| \cdot \|R^1\| \cdots \|R^n\| = \|T\|_{\mathcal{D}_r^*(nM)} \cdot (1 + \varepsilon).$$

Following these steps backwards, we obtain for each factorization of  $T$  as in (5), a representation of  $T$  as in equation (4). Therefore, we have

$$\|T\|_{\mathcal{D}_r^*(nM)} = \inf \{ \|T_\lambda\| \cdot \|R^1\| \cdots \|R^n\| : T \text{ factors as in (5)} \}.$$

In [11, Section 4.3], the ideal of  $r$ -integral polynomials is defined. We define in an analogous way the ideal  $\mathcal{I}_r$  of  $r$ -integral multilinear forms. If  $r \geq n$ , we say that  $T \in \mathcal{L}(X_1, \dots, X_n)$  is  **$r$ -integral** if there exist a finite measure space  $(\Omega, \mu)$  and operators  $S_i : X_i \rightarrow L_r(\mu)$  such that  $T = Q_{\mu,r}^n \circ (S_1, \dots, S_n)$ , where  $Q_{\mu,r}^n \in \mathcal{L}^n(L_r(\mu))$  is the integrating  $n$ -linear form  $Q_{\mu,r}^n(f_1, \dots, f_n) = \int_\Omega f_1 \cdots f_n d\mu$ :

$$\begin{array}{ccc} X_1 & \times \cdots \times & X_n & \xrightarrow{T} & \mathbb{C} \\ S_1 \downarrow & & S_n \downarrow & \nearrow & \\ L_r(\mu) & \times \cdots \times & L_r(\mu) & & Q_{\mu,r}^n \end{array}$$

$\mathcal{I}_r$  is a Banach ideal with the  $r$ -integral norm:

$$\|T\|_{\mathcal{I}_r(X_1, \dots, X_n)} = \inf \{ \|S_1\| \cdots \|S_n\| \cdot \|Q_{\mu,r}^n\| : T = Q_{\mu,r}^n \circ (S_1, \dots, S_n) \}.$$

**Lemma 3.3.** *The  $n$ -linear form  $Q_{\mu,r}^n$  belongs to  $\mathcal{D}_r^*$  and  $\|Q_{\mu,r}^n\|_{\mathcal{D}_r^*(nL_r(\mu))} = \|Q_{\mu,r}^n\| = \mu(\Omega)^{1/u}$ , where  $\frac{1}{u} + \frac{n}{r} = 1$ .*

*Proof.* We have to show that  $Q_{\mu,r}^n$  is a continuous linear form on  $(\bigotimes^n L_r(\mu); (\alpha_r^n)^*)$  with norm  $\mu(\Omega)^{1/u}$ . This is shown in [5, Proposition 18.2] for bilinear forms. Their proof is also valid for  $n \geq 3$ .  $\square$

**Corollary 3.4.** *If  $r \geq n$ ,  $\mathcal{I}_r \subset \mathcal{D}_r^*$  and  $\|T\|_{\mathcal{D}_r^*} \leq \|T\|_{\mathcal{I}_r}$  for each  $r$ -integral  $n$ -linear form  $T$ .*

*Proof.* If  $T$  is  $r$ -integral, it can be written as  $T = Q_{\mu,r}^n \circ (S_1, \dots, S_n)$ . Lemma 3.3 implies that  $T \in \mathcal{D}_r^*$  and

$$\|T\|_{\mathcal{D}_r^*} \leq \|Q_{\mu,r}^n\|_{\mathcal{D}_r^*} \cdot \|S_1\| \cdots \|S_n\| = \|Q_{\mu,r}^n\| \cdot \|S_1\| \cdots \|S_n\|.$$

Taking the infimum over all representations of  $T$  we obtain the desired inequality.  $\square$

**Theorem 3.5.** For  $r \geq n$ , we have  $\mathcal{D}_r^* \stackrel{1}{=} \mathcal{I}_r^{max}$ .

*Proof.* It is enough to show, for  $M_1, \dots, M_n \in \text{FIN}$ , that  $\|T\|_{\mathcal{I}_r(M_1, \dots, M_n)} = \|T\|_{\mathcal{D}_r^*(M_1, \dots, M_n)}$ . One inequality is given in Corollary 3.4. For the other one, we factor  $T \in \mathcal{D}_r^*(M_1, \dots, M_n)$  as

$$\begin{array}{ccccccc} M_1 & \times & \cdots & \times & M_n & \xrightarrow{T} & \mathbb{C} \\ R^1 \downarrow & & & & R^n \downarrow & \nearrow & \\ \ell_r^N & \times & \cdots & \times & \ell_r^N & & T_\lambda \end{array}$$

Let us show now that  $T_\lambda$  can be factored as

$$\begin{array}{ccccccc} \ell_r^N & \times & \cdots & \times & \ell_r^N & \xrightarrow{T_\lambda} & \mathbb{C} \\ J \downarrow & & & & J \downarrow & \nearrow & \\ L_r(\mu) & \times & \cdots & \times & L_r(\mu) & & Q_{\mu, r}^n \end{array}$$

with  $\|Q_{\mu, r}^n\| \cdot \|J\|^n \leq \|T_\lambda\|$ . Since  $T_\lambda$  factors through  $T_{|\lambda|}$ , we can assume that  $\lambda(k) \geq 0$  for each  $k$ . Let  $(\Omega, \mu)$  a measure space such that  $\Omega$  can be splitted as a disjoint union of subsets  $A_1, \dots, A_n$  with  $\mu(A_k) = \lambda(k)^u$ . Let  $J : \ell_r^N \rightarrow L_r(\mu)$  be defined as

$$J(x) = \sum_{k=1}^N x(k) \cdot \lambda(k)^{\frac{1-u}{n}} \cdot \chi_{A_k}.$$

Simple computations show that  $\|J\| = 1$  and  $Q_{\mu, r}^n \circ (J, \dots, J) = T_\lambda$ . Since  $\|Q_{\mu, r}^n\| = \mu(\Omega)^{1/u} = \ell_u(\lambda(k))$  and  $\|T_\lambda\| = \ell_u(\lambda(k))$ . This completes the proof.  $\square$

In [19, Section 22.4], it is shown that absolutely  $r$ -summing operators satisfy  $\lambda(\Pi_r, p, q) + \lambda(\Pi_r^*, q, p) = 1$ . By [14, Corollary 1] (which is analogous to Corollary 2.6), this means that  $\ell(\Pi_r, p, q) \subset \ell_{1/\lambda(\Pi_r, p, q) + \varepsilon}$  for all  $\varepsilon > 0$ . Now, by Proposition 1.2,  $\ell_n(\mathcal{D}_r, p) \subset \ell_{1/\lambda_n(\mathcal{D}_r, p) + \varepsilon}$  for all  $\varepsilon > 0$ . Consequently, by Corollary 2.6 and the remark following it, we have that  $\lambda_n(\mathcal{D}_r; p) + \lambda_n(\mathcal{D}_r^*; p') = 1$ . So we have, for  $r \geq n$ ,

$$\lambda_n(\mathcal{I}_r; p) = \lambda_n(\mathcal{D}_r^*; p) = 1 - \lambda_n(\mathcal{D}_r; p'). \quad (6)$$

We can use now the results of this sections to obtain some properties of the ideal of  $r$ -integral  $n$ -linear forms. For  $n = 2$ , the ideal of  $r$ -integral bilinear forms is maximal [11, 4.4]. Then, by Theorem 3.5, it is the adjoint of the ideal of  $r$ -dominated bilinear forms. By [5] (see also [4]),  $r$ -dominated and 2-dominated bilinear forms coincide for all  $r \geq 2$ . Thus, the same holds for  $r$ -integral bilinear forms:  $\mathcal{I}_r(^2X) = \mathcal{I}_2(^2X)$  for all Banach space  $X$  and all  $r \geq 2$ . This result is not longer true for  $n \geq 3$ . In fact, from [4, Proposition 2.6] and equality 6 we have:

**Corollary 3.6.** Let  $n \geq 3$ . Given  $r \geq n$ , there exists  $p$  such that, for any  $s > r$ , there are diagonal  $s$ -integral  $n$ -linear forms on  $\ell_p$  which are not  $r$ -integral.

## 4 Multiple 1-summing forms

The  $r$ -dominated multilinear mappings generalize the absolutely  $r$ -summing by preserving the Pietsch's Domination Theorem. This happened to be too restrictive a class, in the sense that many other interesting properties of summing operators were not preserved by  $r$ -dominated. This led to the multiple summing multilinear operators. These have been introduced independently by M. Matos [16] and F. Bombal, D. Pérez-García and I. Villanueva [3]. A multilinear operator  $T \in \mathcal{L}(X_1, \dots, X_n; Y)$  is **multiple  $r$ -summing** ( $T \in \Pi_r(X_1, \dots, X_n; Y)$ ) if there exists  $C > 0$  such that for every choice of finite sequences  $(x_j^i) \subseteq X_j$  the following holds

$$\left( \sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(x_1^{i_1}, \dots, x_n^{i_n})\|^r \right)^{\frac{1}{r}} \leq C \cdot w_r((x_1^{i_1})_{i_1=1}^{m_1}) \cdots w_r((x_n^{i_n})_{i_n=1}^{m_n}).$$

The least of such constants  $C$  is called the **multiple  $p$ -summing norm** and denoted  $\|T\|_{\Pi_r(X_1, \dots, X_n; Y)}$ .

A. Defant and D. Pérez-García used multiple 1-summing in [6] to show that its associated tensor norm preserves unconditionality. Some other properties of the norm were also proved and applied in [6, Section 6] to compute the limit order for bilinear multiple 1-summing operators. Their result can be written with our notation as

$$\lambda_2(\Pi_1; p) = \begin{cases} \frac{1}{p'} & \text{if } 2 \leq p \\ \frac{3}{2} - \frac{2}{p} & \text{if } \frac{4}{3} \leq p < 2 \\ 0 & \text{if } 1 \leq p < \frac{4}{3} \end{cases}$$

Our aim is now to compute the limit order of  $\Pi_1$  for higher  $n$ . In fact, what we do is to compute the  $\Pi_1$ -norm of  $\Phi_N : \ell_p^N \times \cdots \times \ell_p^N \rightarrow \mathbb{C}$ .

Let us begin by considering the case  $p \leq 2$ . We follow the steps of [6, Section 6]. First of all, if  $T \in \mathcal{L}(^n \ell_2^N)$  then

$$\|T\|_{\Pi_1(^n \ell_2^N)} \asymp \left( \sum_{i_1, \dots, i_n=1}^N |T(e_{i_1}, \dots, e_{i_n})|^2 \right)^{1/2} \quad (7)$$

(see [18, Theorem 4.2], also [6, Theorem 5.1]). On the other hand, by [6, Theorem 5.2], if  $X$  has 1-unconditional basis, has cotype 2 and  $\dim X = N$ , then for  $S \in \mathcal{L}(^n X)$  we have

$$\|S\|_{\Pi_1(^n X)} \asymp \sup_{\sigma_j} \|S \circ (D_{\sigma_1}, \dots, D_{\sigma_n})\|_{\Pi_1(^n \ell_2^N)}, \quad (8)$$

where  $D_{\sigma_j} : \ell_2^N \rightarrow X$  are norm-one diagonal operators.

Applying (7) and (8) to  $\Phi_N$  we obtain

$$\begin{aligned} \|\Phi_N\|_{\Pi_1(n\ell_p^N)} &\asymp \sup_{\sigma_j} \left( \sum_{k_1, \dots, k_n=1}^N |\sigma_1(k_1) \cdots \sigma_n(k_n) \Phi_N(e_{k_1}, \dots, e_{k_n})|^2 \right)^{1/2} \\ &= \sup_{\sigma_j} \left( \sum_{k=1}^N |\sigma_1(k) \cdots \sigma_n(k)|^2 \right)^{1/2} \end{aligned}$$

where the supremum is taken over all  $\sigma_j$  such that  $D_{\sigma_j} : \ell_2^N \rightarrow \ell_p^N$ ,  $j = 1, \dots, n$  are norm-one operators. Note that  $\|D_{\sigma_j}\| = \|\sigma_j\|_{\ell_r^N}$ , where  $\frac{1}{r} = \frac{1}{2} - \frac{1}{p}$ . If  $r \geq n$ , we have

$$\begin{aligned} \|\Phi_N\|_{\Pi_1(n\ell_p^N)} &\asymp \sup \left\{ \left( \sum_{k=1}^N |\sigma_1(k) \cdots \sigma_n(k)|^2 \right)^{1/2} : \sigma_j \in B_{\ell_r^N} \right\} \\ &= \sup \left\{ \left( \sum_{k=1}^N |\sigma(k)|^2 \right)^{1/2} : \sigma \in B_{\ell_{r/n}^N} \right\} \\ &= \|id : \ell_{r/n}^N \rightarrow \ell_2^N\| = \begin{cases} 1 & \text{if } 1 \leq \frac{r}{n} \leq 2 \\ N^{\frac{1}{2} - \frac{r}{n}} & \text{if } 2 < \frac{r}{n} \end{cases} \end{aligned}$$

If  $r < n$ , with the same procedure we obtain  $\|\Phi_N\|_{\Pi_1(n\ell_p^N)} \asymp 1$ . Since the reverse inequality is always true, we also have  $\|\Phi_N\|_{\Pi_1(n\ell_p^N)} \asymp 1$  for this case. This gives:

$$\|\Phi_N\|_{\Pi_1(n\ell_p^N)} \asymp \begin{cases} 1 & \text{if } 1 \leq p \leq \frac{2n}{n+1} \\ N^{\frac{n+1}{2} - \frac{n}{p}} & \text{if } \frac{2n}{n+1} \leq p \leq 2. \end{cases}$$

We now consider  $p > 2$ . Let us see that in this case

$$\max\{\sqrt{N}, N^{1 - \frac{n-1}{p}} (\log N)^{1/p'}\} \prec \|\Phi_N\|_{\Pi_1(n\ell_p^N)} \prec N^{\frac{n+1}{2} - \frac{n}{p}}.$$

To get the lower bound, first note that we can factor

$$\begin{array}{ccccccc} \ell_2^N & \times & \cdots & \times & \ell_2^N & \xrightarrow{\Phi_N} & \mathbb{C} \\ i \downarrow & & & & i \downarrow & \nearrow_{\Phi_N} & \\ \ell_p^N & \times & \cdots & \times & \ell_p^N & & \end{array}$$

This factorization together with (7) give  $\sqrt{N} = \|\Phi_N\|_{\Pi_1(n\ell_2^N)} \leq \|\Phi_N\|_{\Pi_1(n\ell_p^N)}$ .

We complete the lower bound by induction on  $n$ . By [6, Lemma 3.4] we have the isometry  $\Pi_1(n\ell_p^N; \mathbb{C}) \stackrel{1}{=} \Pi_1(n-1\ell_p^N; \ell_{p'}^N)$ , denoted by  $T \leftrightarrow \tilde{T}$ . If  $n = 2$ , then  $\tilde{\Phi}_N = id : \ell_p^N \rightarrow \ell_{p'}^N$ . Therefore,  $\|\Phi_N\|_{\Pi_1(2\ell_p^N)} = \|id\|_{\Pi_1(\ell_p^N; \ell_{p'}^N)} \asymp (N \log N)^{1/p'}$ , by [19, 22.4.11].

Let us now consider  $\Sigma_N : \ell_{p'}^N \rightarrow \mathbb{C}$  given by  $z \rightsquigarrow \sum_{k=1}^N z(k)$  and  $\Psi_N = \Sigma_N \circ \tilde{\Phi}_N : \underbrace{\ell_p^N \times \cdots \times \ell_p^N}_{(n-1) \text{ times}} \rightarrow \ell_{p'}^N \rightarrow \mathbb{C}$ . By the induction hypothesis

$$N^{1-\frac{n-2}{p}} (\log N)^{1/p'} \prec \|\Psi_N\|_{\Pi_1(n-1)\ell_p^N} \leq \|\Phi_N\|_{\Pi_1(n)\ell_p^N} \|\Sigma_N\|.$$

Now, since  $\|\Sigma_N\| = N^{\frac{1}{p}}$ , we have the desired lower bound.

To get the upper bound, let us factor  $\Phi_N$  in the following way

$$\begin{array}{ccccccc} \ell_p^N & \times & \cdots & \times & \ell_p^N & \xrightarrow{\Phi_N} & \mathbb{C} \\ \downarrow & & & & \downarrow & \nearrow & \\ \ell_2^N & \times & \cdots & \times & \ell_2^N & & \end{array}$$

Hence

$$\begin{aligned} \|\Phi_N\|_{\Pi_1(n)\ell_p^N} &\leq \|id : \ell_p^N \rightarrow \ell_2^N\|^n \|\Phi_N\|_{\Pi_1(n)\ell_2^N} \\ &\prec (N^{\frac{1}{2}-\frac{1}{p}})^n \sqrt{N} = N^{\frac{n+1}{2}-\frac{n}{p}}. \end{aligned}$$

This altogether gives the following situation

$$\begin{aligned} \|\Phi_N\|_{\Pi_1(n)\ell_p^N} &\prec 1 && \text{if } 1 \leq p \leq \frac{2n}{n+1} \\ \|\Phi_N\|_{\Pi_1(n)\ell_p^N} &\prec N^{\frac{n+1}{2}-\frac{n}{p}} && \text{if } \frac{2n}{n+1} \leq p \leq 2 \\ \max\{\sqrt{N}, N^{1-\frac{n-1}{p}} (\log N)^{1/p'}\} &\prec \|\Phi_N\|_{\Pi_1(n)\ell_p^N} &\prec N^{\frac{n+1}{2}-\frac{n}{p}} && \text{if } 2 \leq p. \end{aligned}$$

We can reformulate this results in terms of limit orders and defects:

For  $p \leq 2$ :

$$\lambda_n(\Pi_1; p) = \begin{cases} 0 & \text{if } 1 \leq p \leq \frac{2n}{n+1} \\ \frac{n+1}{2} - \frac{n}{p} & \text{if } \frac{2n}{n+1} \leq p \leq 2 \end{cases}$$

and  $d_n(\Pi_1, p) = 0$ .

For  $p > 2$ :

$$\max\left\{\frac{1}{2}, 1 - \frac{n-1}{p}\right\} \leq \lambda_n(\Pi_1; p) \leq \min\left\{\frac{n+1}{2} - \frac{n}{p}, 1\right\}$$

and  $d_n(\Pi_1, p) \leq \frac{n-1}{2} - \frac{1}{p}$ .

## 5 The diagonal of a multilinear form

Throughout this section,  $X_i$  will be Banach spaces with unconditional basis  $\{e_j^i\}_j$  ( $i = 1, \dots, n$ ). We define the mapping  $D : \mathcal{L}(X_1, \dots, X_n) \rightarrow \mathcal{L}(X_1, \dots, X_n)$  given by

$$D(T)(x_1, \dots, x_n) = \sum_{j=1}^{\infty} T(e_j^1, \dots, e_j^n) x_1(j) \cdots x_n(j).$$



Note that  $D(T)$  is the diagonal  $n$ -linear form given by the diagonal of  $T$ . The linear mapping  $D$  is well defined and continuous [7, Proposition 1.3]. Let us show now that it preserves some ideals of multilinear forms.

**Proposition 5.1.** *Let  $\beta$  be a tensor norm of order  $n$ . If  $T \in \mathcal{L}(X_1, \dots, X_n)$  is  $\beta$ -continuous (i.e.,  $T \in (\bigotimes_{i=1}^n X_i; \beta)'$ ), then  $D(T)$  is also  $\beta$ -continuous.*

*Proof.* For  $0 \leq t \leq 1$  and  $i = 1, \dots, n$ , we define  $\Lambda_t^i : X_i \rightarrow X_i$  by

$$\Lambda_t^i(x) = \sum_{j=1}^{\infty} x(j)r_j(t)e_j^i,$$

where  $\{r_j\}_j$  are the generalized  $n$ -Rademacher functions [2, Section 1]. By the unconditionality of the basis,  $\Lambda_t^i$  is continuous and  $\|\Lambda_t^i\| \leq 2K_i$  (being  $K_i$  the unconditionality constant of the basis).

Let  $s \in \bigotimes_{i=1}^n X_i$ ,  $s = \sum_{k=1}^M x_k^1 \otimes \dots \otimes x_k^n$ . We have

$$\begin{aligned} |D(T)(s)| &= \left| \sum_{k=1}^M D(T)(x_k^1, \dots, x_k^n) \right| \\ &= \left| \int_0^1 \sum_{k=1}^M T(\Lambda_t^1(x_k^1), \dots, \Lambda_t^n(x_k^n)) dt \right| \\ &= \left| \int_0^1 T((\Lambda_t^1 \otimes \dots \otimes \Lambda_t^n)(s)) dt \right| \\ &\leq \|T\|_{(\bigotimes_{i=1}^n X_i; \beta)'} \cdot \|\Lambda_t^1\| \cdots \|\Lambda_t^n\| \cdot \|s\|_{\beta} \\ &\leq \|T\|_{(\bigotimes_{i=1}^n X_i; \beta)'} \cdot 2^n K_1 \cdots K_n \cdot \|s\|_{\beta} \end{aligned}$$

□

**Corollary 5.2.** *If  $\mathfrak{A}$  is a maximal ideal of  $n$ -linear forms, then  $D : \mathfrak{A}(X_1, \dots, X_n) \rightarrow \mathfrak{A}(X_1, \dots, X_n)$  is well defined and continuous.*

In particular, the ideals of integral, extendible,  $r$ -dominated ( $r \geq n$ ) and multiple  $r$ -summing  $n$ -linear forms are preserved by  $D$ . An example of an ideal which is not maximal but is preserved by  $D$  is given in the following:

**Proposition 5.3.** *If  $T \in \mathcal{L}(X_1, \dots, X_n)$  is weakly sequentially continuous, then so is  $D(T)$*

*Proof.* Let  $(x_k^i)_k \subseteq X_i$  be weakly convergent to  $x^i \in X_i$ . With the notation of the proof of Proposition 5.1, we have

$$D(T)(x_k^1, \dots, x_k^n) = \int_0^1 T(\Lambda_t^1(x_k^1), \dots, \Lambda_t^n(x_k^n)) dt.$$

Since each  $\Lambda_t^i$  is linear and  $T$  is weakly sequentially continuous,  $T(\Lambda_t^1(x_k^1), \dots, \Lambda_t^n(x_k^n))$  converges to  $T(\Lambda_t^1(x^1), \dots, \Lambda_t^n(x^n))$  for every  $t \in [0, 1]$ . Now the result follows from the dominated convergence theorem. □

However, not every ideal is preserved by  $D$ :

**Example 5.4.** *The ideals of nuclear, approximable and weakly continuous on bounded sets multilinear forms are not preserved by  $D$ :*

Let  $T \in \mathcal{L}({}^n\ell_1)$  be given by

$$T(x^1, \dots, x^n) = \left( \sum_{j=1}^{\infty} x^1(j) \right) \cdots \left( \sum_{j=1}^{\infty} x^n(j) \right).$$

Clearly,  $T$  is a finite type  $n$ -linear form. Now,

$$D(T)(x^1, \dots, x^n) = \sum_{j=1}^{\infty} x^1(j) \cdots x^n(j)$$

which is not weakly continuous on bounded sets (see, for example, [8, Proposition 2.6]). Hence it is neither nuclear, nor approximable.

From the above results about the operator  $D$  and the definitions of limit order and defect, we can obtain some information about the diagonal of any  $n$ -linear form belonging to certain ideal.

**Proposition 5.5.** *If  $\mathfrak{A}$  is a maximal ideal of  $n$ -linear forms (or any ideal of  $n$ -linear forms preserved by  $D$ ) and  $T \in \mathfrak{A}({}^n\ell_p)$ , then, for every  $\varepsilon > 0$ ,*

$$(T(e_j, \dots, e_j))_j \in \ell_{r+\varepsilon}$$

where  $r = \frac{1}{\lambda_n(\mathfrak{A};p) - d_n(\mathfrak{A};p)}$  and  $(e_j)_j$  is the canonical basis of  $\ell_p$ .

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